

Knots & k -width

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Abstract

We investigate several natural integer invariants of curves in \mathbb{R}^3 and explore their values on isotopy classes of curves.

1 Introduction

In this paper we introduce the notion of k -width for a knot or link in \mathbb{R}^3 , where k is an integer between 1 and 4. These widths provide increasingly detailed information, as k increases, on the intersections of a curve with flat planes and round spheres in \mathbb{R}^3 . We examine properties of curves that minimize k -width within their isotopy class.

The notion of k -width is motivated by considerations in the theory of index- k minimal submanifolds. Our notion of k -width does not appear to have been studied before for $k > 1$. After introducing k -width, we explore properties of 2-width in some detail. We show that only finitely many knots have 2-width less than c , for any positive constant c . Thus, like crossing number, 2-width gives a way to order all knots in terms of increasing complexity. We also show that only the unknot and the trefoil have 2-width less than 10.

We also relate the 2-width of a curve to its curvature. Milnor and Fary showed that if a smooth curve γ has total curvature less than 4π then γ is unknotted [8] [4]. The same argument shows that if the total curvature of γ

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is less than $2n\pi$ then there is a direction relative to which γ has at most n maxima. As a consequence, its bridge number is at most n . The converse is false. A curve can be deformed to have arbitrarily large curvature without changing the bridge number. We show here that if an immersed plane curve γ has 2-width n then it has total curvature at most $2\pi n^{3/2}$. We prove that if a curve γ in \mathbb{R}^3 has 2-width n , then some planar projection of γ has total curvature at most $2\pi n^{3/2}$. This can be viewed in contrast to the Fary-Milnor Theorem. While small bridge number does not imply that some projection has small total curvature, small 2-thickness does imply this.

In Section 2 we introduce k -width for curves in \mathbb{R}^3 . In Section 3 we show that there are only a finite number of knot types with 2-width bounded by a given constant and in Section 4 we look at the knots with the lowest 2-width. In Section 5 we establish a connection between 2-width and the total curvature of a curve. Small 2-width implies that the total curvature of some projection of a curve is small. Finally in Section 6 we discuss a far more general extension of the notion of width, involving the intersection between a submanifold of a manifold and a group of submanifolds.

2 Width for curves in \mathbb{R}^3

Combinatorial knot invariants computed by counting intersections of curves with planes originate with Schubert's bridge number [9], and also include Kuiper's superbridge number [7] and Gabai's thin position [6]. Let γ be a smooth curve in \mathbb{R}^3 . The 1-width of γ is found in [6]. It is computed by counting the sum of a curve's intersections with a finite collection of horizontal planes in \mathbb{R}^3 , one plane being chosen between each pair of horizontal tangents of γ . We present natural definitions of 2, 3 and 4-parameter families of planes and spheres that give interesting width invariants. We will describe these in \mathbb{R}^3 , but they are equally natural in \mathbb{S}^3 .

Definition. Consider the set X_2 of planes in \mathbb{R}^3 perpendicular to the xy -plane. This set is in 1-1 correspondence with the set of straight lines in the plane, which is diffeomorphic to a punctured $\mathbb{R}P^2$, or equivalently to an open Mobius band. Assume that a smooth curve γ has been perturbed slightly, so that it has no tangent lines parallel to the z -axis and its projection onto the xy -plane is in general position, with finitely many transverse double points. Moreover we can assume that no plane in X_2 is tangent to γ at more than two points. Set two planes in X_2 equivalent if they are connected by a path

in X_2 consisting of planes transverse to γ . The *width* of a plane transverse to γ is the number of intersection points of the plane with γ . Equivalent planes intersect γ in the same number of points. Summing over representatives of each equivalence class of planes transverse to γ gives an integer $w_2(\gamma)$ called the *2-width* of γ . A curve γ is *2-width minimizing* if it minimizes 2-width in its isotopy class. In the next sections we will explore some properties of 2-width. The 2-width $w_2(K)$ of a knot K is the 2-width of a 2-width minimizing representative of the knot.

There is a unique plane in X_2 which is tangent at every point on the curve γ , by our assumption that no tangent line is parallel to the z -axis. It follows that the set of non-transverse planes forms a singular curve λ in the open Mobius band X_2 . Double tangencies of γ with planes in X_2 give double points of λ and inflection points of the planar projection of γ give rise to cusps on λ . Hence the 2-width is finite, since the number of regions in X_2 of planes transverse to γ is the same as the finite number of complementary regions of λ in the open Mobius band X_2 .

It is possible to define 2-width working entirely with the projection $\pi(\gamma)$ of γ onto the xy -plane, since changes in its z -coordinate do not affect the intersections of γ with a plane in X_2 . To define 3-width and 4-width however, we are obliged to work with 3-dimensional representations of a knot.

Definition. Let X_3 be the set of all planes in \mathbb{R}^3 . X_3 is diffeomorphic to a once punctured \mathbb{RP}^3 . Two planes in X_3 are said to be equivalent if they are connected through a path of planes transverse to γ . The *width* of a plane transverse to γ is the number of intersection points of the plane with γ . The *3-width* $w_3(\gamma)$ of γ is the sum of the widths over all equivalence classes of planes transverse to γ . A curve is *3-width minimizing* if it minimizes 3-width in its isotopy class.

At any point x of γ , there is a circle of planes touching γ at x . So the set of planes that are not transverse to γ forms a singular torus in X_3 . Double curves of this torus are double tangencies of γ and triple points are triple tangencies. It is then clear that 3-width is finite, as it is a sum of intersection numbers of planes, one for each of the complementary regions of this singular torus.

A related but distinct invariant is the superbridge index introduced by N. Kuiper [7]. In our point of view, bridge number is a 1-parameter and superbridge index is a 3-parameter invariant. It is possible to define a 2-parameter and 4-parameter bridge index as well, though these do not appear to have been investigated. In [7] it is shown that there are infinitely many

knots with superbridge index bounded by 4. In fact, Kuiper showed that for all odd q , an appropriate embedding of the $(2, q)$ -torus knot has superbridge index 4. These superbridge indices can be realized by positioning the $(2, q)$ torus knot as satellites of a “baseball” curve on the 2-sphere, a curve that intersects any plane in at most four points. When positioned on a small radius torus around this baseball curve, the $(2, q)$ torus knot intersects any flat plane in at most 8 points. Nevertheless, the 3-width of these torus knots does not appear to be bounded.

A 3-width minimizing curve gives one approach to finding an “optimal” imbedding of a knot. Note however that the 3-width of a curve is preserved by affine maps of \mathbb{R}^3 , and these can radically change the shape of a curve. Similarly the 2-width of a curve is preserved by affine maps of \mathbb{R}^2 that preserve the set of planes X_2 .

Finally we consider a 4-parameter width.

Definition. Let X_4 be the set of all flat planes and round 2-spheres in \mathbb{R}^3 . The topology on X_4 is obtained by considering the space of round spheres on the 3-sphere, identified to X_4 via stereographic projection. X_4 is diffeomorphic to a once punctured \mathbb{RP}^4 . Define the width of a sphere or plane transverse to γ to be the number of its intersection points with γ . Two spheres or planes are equivalent if they can be connected by a path in X_4 consisting of planes and spheres with no tangencies to γ . The *4-width* of γ is the sum of the widths over all equivalence classes of planes and spheres transverse to γ . A curve is *4-width minimizing* if it minimizes 4-width over all smooth curves in its isotopy class.

As before, one can show that the set of non-transverse spheres and planes is a singular 3-manifold, the image of a product of a Mobius band and a circle. Addition of widths for the finitely many complementary regions gives the 4-width.

Like the knot energy studied in [5], w_4 is a conformal invariant of γ . Unlike knot energies however, w_4 is integer valued.

Lemma 2.1. *If γ' is the image of γ by a conformal map of \mathbb{R}^3 , then $w_4(\gamma') = w_4(\gamma)$.*

Proof. There is a width preserving 1-1 correspondence between the equivalence classes for γ and those for γ' . \square

We can define higher order widths by considering intersections with quartics and other families of surfaces. See Section 6 for some very general extensions. The main results of this paper concern 2-width.

3 The number of knots with bounded 2-width

Knot tables are usually arranged to reflect increasing crossing number. This is feasible because for any constant n , only finitely many knots have crossing number less than n . This property does not hold for other common invariants, such as unknotting number, tunnel number and bridge number. While 1-width shares this shortcoming, we prove below a finiteness result for the 2-width invariant.

Theorem 3.1. *For any constant n , only finitely many knots in \mathbb{R}^3 have 2-width less than or equal to n .*

Proof. Let $\gamma \subset \mathbb{R}^3$ be a generic embedded curve. We require additional properties to those in the definition of 2-width, where it was assumed that γ has no vertical tangencies and the projection to the horizontal xy -plane is in general position. For convenience, we will specify later what we mean by generic.

Recall that the set X_2 of planes perpendicular to the xy -plane in \mathbb{R}^3 is diffeomorphic to a punctured projective plane. We will not distinguish in what follows between a point of X_2 and the corresponding plane in \mathbb{R}^3 . We associate to γ a certain graph α in X_2 . Using the projection map $\pi : \mathbb{R}^3 \rightarrow \mathbb{R}^2$, we project γ to a planar curve $\pi(\gamma)$ in the xy -plane. To each point in $\pi(\gamma)$ we associate the tangent line at that point. The set of tangent lines gives a graph α in X_2 , which we call the *graphic* of γ in X_2 . (Each plane in X_2 corresponds to such a line in the xy -plane). The graph α is the image of a curve that is immersed in X_2 except at a finite number of cusps. The cusps are the images of inflection points of $\pi(\gamma)$. A finite number of planes in X_2 are tangent to γ at two points. Such a *double tangent plane* corresponds to an order four vertex in α . Let v be the number of these vertices.

Let r count the number of regions of the complement of α in X_2 . These regions consist of f simply-connected faces, one non-compact annulus and possibly one Mobius band. Two points in a given region R_i correspond to planes in X_2 in the same equivalence class, so we can assign an integer $w_2(R_i)$ to that region, equal to the number of times a representative plane intersects γ . The 2-width of γ is the sum of the integers $w_2(R_i)$ over all the regions.

The double tangent planes of γ are divided into two types. A double tangent plane is said to be *exterior* if it does not separate small neighborhoods of the two tangent points and *interior* if it does separate them. Let t be the number of exterior double tangent planes of γ and s the number of interior

double tangent planes. Let i be the number of inflection points and let c be the number of crossings of $\pi(\gamma)$. An inflection point corresponds to a point where γ is tangent to a vertical plane at a point and the plane separates any neighborhood of the point of tangency. A crossing point of $\pi(\gamma)$ is a point where a vertical line meets γ twice.

A theorem of Fabricius-Bjerre [3] provides the tool needed to link properties of γ to the graph α . Fabricius-Bjerre considered the relation between the numbers of double tangent lines, crossing points and inflection points of a plane curve δ . He established the following result [3].

Theorem 3.2 (Fabricius-Bjerre). *Let δ be a generic plane curve with t exterior double tangent lines, s interior double tangent lines, i inflection points and c crossings. Then $c + \frac{1}{2}i = t - s$.*

In this context, *generic* curves are smooth curves in general position, with finite values for each of the quantities in Theorem 3.2. For present purposes we can take a generic curve in \mathbb{R}^3 to be one whose projection is generic in this sense.

Lemma 3.3. $c \leq v$.

Proof. Double tangent planes in X_2 correspond to valence-4 vertices of the graph α , so the number of such vertices v of α equals $s + t$. Applying Fabricius-Bjerre's Theorem, we obtain

$$c \leq c + \frac{1}{2}i = t - s \leq t + s = v.$$

□

The next lemma relates v to the number of simply connected faces f in the complement of α . While α may have cusps, they do not play a role in this computation, so we ignore them.

Lemma 3.4. $v = f$ and either $r = f + 1$ or $r = f + 2$. The latter occurs when there is a region in the graphic homeomorphic to a Mobius band.

Proof. There is one region in the complement of α homeomorphic to a half-open annulus. There may be one region homeomorphic to a Mobius band. Annular and Mobius band regions do not contribute to the Euler characteristic computation. Let e count the number of edges connecting valence four vertices. The Euler characteristic of X_2 is zero, obtained by

taking the sum, giving $v - e + f = 0$. Since vertices are of valence four, we have $e = 2v$, and so $f = v$ as claimed. Depending on whether a Mobius band component exists in the complement of α , we have $r = f + 1$ or $r = f + 2$. \square

Now let R_1, R_2, \dots, R_r denote the regions of the graph α , with associated widths w_1, w_2, \dots, w_r respectively. Each w_i is a non-negative even integer, and their sum is n . At most one of the w_i 's equals zero, since there is a single equivalence class of planes in \mathbb{R}^3 disjoint from γ . Thus each contributes at least 2 to the total width of γ and the total width $w_2(\gamma)$ satisfies $w_2(\gamma) \geq 2r > 2f = 2v \geq 2c$.

So the crossing number of γ is bounded above by half of its 2-width. Since only finitely many knots have crossing number less than a given integer, the theorem follows. \square

4 Bounds on 2-width

Definition. A knot projection is *positively curved* if it contains no inflection points.

Lemma 4.1. *Every curve in \mathbb{R}^3 can be isotoped to have a positively curved projection.*

Proof. Every curve γ is isotopic to a curve with a braid presentation [2]. Such a presentation is parametrized in cylindrical coordinates by a curve $\gamma(s) = (r(s), \theta(s), z(s))$ with $\theta'(s) > 0$ and $r(s) > 0$. By scaling we can assume $0 < r(s) < 1$. An isotopy $\gamma_t(s) = (t + (1 - t)r(s), \theta(s), z(s))$, $0 \leq t \leq 1 - \epsilon$ takes γ to a curve in a 2ϵ -neighborhood of the unit radius cylinder $\{r = 1\}$. As $\epsilon \rightarrow 0$, $\pi(\gamma)$ smoothly converges to a cover of the unit circle. Thus for ϵ sufficiently small, $\pi(\gamma_{1-\epsilon})$ has all curvatures positive. \square

We can apply this to get an upper bound on the 2-width of a knot.

Proposition 4.2. *Let c be the minimal crossing number of a braid projection of a knot K . Then $w_2(K) \leq (c + 1)(c + 2)$,*

Proof. Choose γ representing K so that $\pi(\gamma)$ has minimal crossing number over all braid representations of K . Further isotop γ so that its projection $\pi(\gamma)$ is positively curved as in Lemma 4.1. Then $\pi(\gamma)$ has no inflection points and no internal double tangencies. Therefore its crossing number c equals the number of external double tangencies t , which in turn equals the number of vertices v in the graphic of $\pi(\gamma)$. From Lemma 3.4 we know that the number

of regions in the graphic r satisfies $c + 1 \leq r \leq c + 2$. Each region has width a non-negative even integer, and one of the regions has width zero. Regions in the graphic with a common edge have width that differs by 2. So the sum of the widths is at most $2 + 4 + 6 + \dots + 2(c + 1) = (c + 2)(c + 1)$. \square

We can obtain a lower bound in terms of the number of intersections between the curve and a vertical plane.

Proposition 4.3. *If a plane in X_2 intersects γ in $2n$ points then $w_2(\gamma) \geq n(n + 1)$*

Proof. Some plane in X_2 intersects γ in zero points. Each region in the graphic has width a positive even integer, and at least one region has width $2n$. Regions in the graphic with a common edge have width that differs by 2, so there are regions in the graphic of widths $2, 4, \dots, 2n$ and the total width is at least $2 + 4 + 6 + \dots + 2n = n(n + 1)$. \square

Theorem 4.4. *The only knots with 2-width less than or equal to 10 are the trefoil and the unknot.*

Proof. Suppose $w_2(\gamma) \leq 10$. If γ meets a plane in X_2 in at least 6 points then Lemma 4.3 implies that its width is at least 12. So we can assume that γ meets any plane in X_2 in at most 4 points.

Call a crossing of $\pi(\gamma)$ *exterior* if it meets the unbounded region of the complement of $\pi(\gamma)$ in the xy -plane. Otherwise call it *interior*. A ray in the xy -plane from a point near an interior crossing to infinity must cross $\pi(\gamma)$ at least once. If γ is knotted then $\pi(\gamma)$ has an exterior crossing. Suppose $\pi(\gamma)$ also has an interior crossing. Then a line in the xy -plane passing very close to both an interior and an exterior crossing meets $\pi(\gamma)$ twice near each crossing and at least once more in passing from a point near the interior crossing to infinity. Since the intersection number with $\pi(\gamma)$ is even, the intersection consists of at least six points. But then γ meets a plane in X_2 in six points and $w_2(\gamma) \geq 12$. So we can assume that all crossings of $\pi(\gamma)$ are exterior.

Color the complementary regions to $\pi(\gamma)$ black and white, with the unbounded region white and regions with a common edge having different colors. Rays based at points in a white region intersect $\pi(\gamma)$ in an even number of points, while rays based in a black region intersect $\pi(\gamma)$ an odd number of times. If there are no bounded white regions then a small neighborhood of any crossing meets the unbounded region in two components and a simple induction argument shows that γ is unknotted, satisfying the conclusion of

the Theorem. Otherwise there is at least one bounded white region. A ray from a bounded white region to infinity meets $\pi(\gamma)$ in at least two points. If there are two distinct bounded white regions, then a line segment connecting a point in each intersects $\pi(\gamma)$ in at least 2 points. Extending this segment to a line by adding two rays gives a line intersecting $\pi(\gamma)$ in at least six points, contradicting our width assumption. So we can assume that $\pi(\gamma)$ has exactly one bounded white region, W .

Let w be an interior point of W . Every line through w intersects $\pi(\gamma)$ in exactly 4 points. So $\pi(\gamma)$ gives a 2-braid projection of γ , with axis the vertical line over w . The number of crossings of a 2-braid knot is odd, and since we assume that γ is not a trefoil or unknot we have that the number of crossings is at least 5. Applying Lemma 3.3 shows that the number of regions in the graphic is 7, with one annular and one Mobius band region. One region has width 4 and five other regions have width at least two, giving that the 2-width of γ is at least 14, contradicting our assumption.

We conclude that γ is a trefoil or an unknot. \square

Theorem 4.5. *The $(2, n)$ -torus knot K has 2-width $2n + 4$.*

Proof. A positively curved 2-braid projection of K has a graphic with $r = n + 2$. One region has width 0, one has width 4 and n have width 2, giving a total width of $2n + 4$. It remains to show no representative of K has smaller 2-width.

Let γ be any representative of K . Since K has n crossings in an alternating projection, the number of crossings of $\pi(\gamma)$ is at least n . Applying Lemmas 3.3 and 3.4 shows that the number of regions in the graphic of $\pi(\gamma)$ is $n + 1$ or $n + 2$. As in the proof of Theorem 4.4 there is at least one interior white region for $\pi(\gamma)$. An interior white region has width at least 4. If there is exactly one then the graphic has a Mobius band region of width 4 and $r = n + 2$. In that case the width is at least $4 + 2n$ as claimed. If there is more than one interior white region, then the total width is at least $4 + 4 + 2(n - 2) = 2n + 4$. The result follows. \square

Remark: Width can be defined and computed for links in the same way. The number of zero-width regions in the graphic associated to a link can be greater than one if the link is split (separated by a plane.) The Hopf link and the 2-component unlink each have $w_2(L) = 8$ and the $(2, 4)$ -torus link has $w_2(L) = 12$.

5 Curvature and 2-width

The *total curvature* of a curve in \mathbb{R}^3 is obtained by integrating the absolute value of the curvature function. Milnor and Fary showed that if a smooth curve γ has total curvature less than 4π then γ is unknotted [8], [4]. Milnor's proof proceeds by finding a direction in \mathbb{R}^3 relative to which the curve has only one maximum and one minimum. The same argument shows that if the total curvature of γ is less than $2n\pi$ then there is a direction relative to which γ has fewer than n maxima. As a consequence, its bridge number is less than n . The converse is false. Curves of bridge number two can have unbounded curvature. In fact any curve can be deformed to have arbitrarily large total curvature without changing its bridge number. So small bridge number does not imply small total curvature. In contrast, we show here that if a curve has small 2-width then its projection onto the associated 2-plane has small total curvature.

We first obtain a result about immersed plane curves. When a curve lies in the xy -plane the notion of 2-width can be restated in terms of its intersections with lines in the plane rather than planes in \mathbb{R}^3 . Lemma 5.1 considers curves immersed in the plane.

Lemma 5.1. *Let γ be a C^2 curve immersed in the plane containing an embedded positively curved arc with total curvature x . Then $w_2(\gamma) > x^2/(2\pi)^2$.*

Proof. An embedded positively curved arc with total curvature x is a spiral that winds monotonically around one of its endpoints, with total angle x . For if the arc did not wind monotonically, it would have to touch a ray starting at the given endpoint on one side. But then it is easy to see that the curvature of the arc must have changed sign and this is a contradiction.

A line through the two endpoints of this spiral intersects γ in at least $[x/\pi] + 1$ points. Assume first that $[x/\pi] + 1$ is even. Then the 2-width of γ is at least

$$([x/\pi] + 1) + ([x/\pi] - 1) + \dots + 2 = ([x/\pi] + 1)([x/\pi] + 2)/4 > x^2/(2\pi)^2.$$

On the other hand, if $[x/\pi] + 1$ is odd, then since any line meets γ in an even number of points, the line through the endpoints of the spiral intersects γ in at least $[x/\pi] + 2$ points and the same argument clearly applies. \square

Theorem 5.2. *If γ is a C^2 curve immersed in the plane with total curvature x then $w_2(\gamma) > Cx^{2/3}$, where $C = 1/(2\pi)^{2/3}$.*

Proof. Let $y = 2c + i$ where c is the number of crossings of γ and i is the number of its inflection points. Now by Theorem 3.2 we have that $2c + i = 2t - 2s \leq 2t + 2s = 2v$ where v counts the vertices in the graphic of γ . Lemma 3.4 implies that $v = f$ and either $r = f + 1$ or $r = f + 2$. So $w_2(\gamma) \geq 2f \geq y$. If $y \geq x^{2/3}/(2\pi)^{2/3}$ and the Theorem follows. So assume $y < x^{2/3}/(2\pi)^{2/3}$.

The crossing and inflection points of γ divide γ into $y = 2c + i$ subarcs, each with no crossing or inflection points. Since γ has total curvature x , one of the subarcs of γ disjoint from its inflection and crossing points is positively curved with total curvature greater than x/y . By Lemma 5.1 we have that

$$w_2(\gamma) > (x/y)^2/(2\pi)^2 = x^2/(2y\pi)^2 > (2\pi)^{4/3}x^2/(2\pi x^{2/3})^2 = x^{2/3}/(2\pi)^{2/3}.$$

□

Theorem 5.2 gives the following corollary for curves in \mathbb{R}^3 .

Corollary 5.3. *If γ is a curve in \mathbb{R}^3 with $w_2(K) \leq n$ then some planar projection of K has total curvature at most $2\pi n^{3/2}$.*

Proof. If a planar projection of γ has curvature greater than $2\pi n^{3/2}$ then Theorem 5.2 implies that $w_2(\gamma) > n$. □

6 Width in higher dimensions and codimensions

We now define width for a general submanifold. We first consider a submanifold Γ and a collection of submanifolds transverse to Γ . We then consider a more general situation where Γ is not necessarily transverse to a family of submanifolds.

Definition 1. Let N and Γ be submanifolds of a manifold M and let G be a k -dimensional group of diffeomorphisms of M . Let X denote the set of submanifolds $\{g(N) : g \in G\}$ that transversely intersect Γ . Define an equivalence relation on X by setting $g_0(N) \sim g_1(N)$ if there is a path from g_0 to g_1 in G such that $g_t(N), 0 \leq t \leq 1$ is in X . So $g_0(N) \sim g_1(N)$ if $g_0(N)$ is isotopic to $g_1(N)$ through submanifolds $g_t(N)$ transverse to Γ .

Equivalent submanifolds $g_0(N)$ and $g_1(N)$ intersect Γ in the same number of points. Pick one representative $g_i(N)$ for each equivalence class. The *width*

w_G of Γ is defined to be the sum over equivalence classes of the number of intersections of $g_i(N)$ and Γ ;

$$w_G(\Gamma) = \sum_i |\Gamma \cap g_i(N)|.$$

We say that Γ is w_G -minimizing if Γ minimizes w_G within its isotopy class.

Example 1. Let M, N, G be \mathbb{R}^3 , the xy -plane and \mathbb{R}^1 respectively. For $g \in \mathbb{R}^1$ let $g((x, y, 0)) = (x, y, g)$. Then G defines a foliation of \mathbb{R}^3 by horizontal planes. A curve Γ is w_G -minimizing if it is in thin position as in [6].

Example 2. Let M, N, G be $\mathbb{S}^3 \times \mathbb{S}^1$, $\mathbb{S}^3 \times \{1\}$, and \mathbb{S}^1 , respectively, and let \mathbb{S}^1 act on M by rotation. A curve γ in M is in thin position relative to w_G if it either lies in a leaf neighborhood with a single maximum and minimum and has width two or is always transverse to the leaves. In the latter case there is a single equivalence class and the homotopy class of γ determines its width, equal to $\omega(\gamma)$, where ω generates $H^1(M; \mathbb{Z})$.

It is not necessary to restrict our attention to transverse intersections. We can obtain still more general measures of intersection complexity by using the following definition.

Definition 2. Let N and Γ be submanifolds of a manifold M and let G be a k -dimensional group of diffeomorphisms acting on M . Let X denote the set of submanifolds $\{g(N) : g \in G\}$ and define an equivalence relation on X by setting $g_0(N) \sim g_1(N)$ if there is a path from g_0 to g_1 in G such that $g_t(N) \cap \Gamma$ is isotopic to $g_0(N) \cap \Gamma$, $0 \leq t \leq 1$. So $g_0(N) \sim g_1(N)$ if $g_0(N)$ is isotopic to $g_1(N)$ through submanifolds whose intersection with Γ is preserved up to isotopy. (Transversal intersection is no longer required.) The width of a submanifold Γ is the intersection number of Γ and $\{g(N) : g \in G\}$ counted with appropriate multiplicity.

Example 3. Let M, N, G be \mathbb{R}^3 , the z -axis and \mathbb{R}^2 respectively. For $g \in \mathbb{R}^2$ let $g((0, 0, z)) = (g, z)$. Then G defines a foliation of \mathbb{R}^3 by vertical lines. Let γ be a smooth curve.

For a generic γ , the equivalence classes of X_G consist of

- Sets of lines that miss γ , one for each complementary component of the projection $\pi(\gamma)$ to the xy -plane.
- One component for each edge of $\pi(\gamma)$.
- One component for each vertex of $\pi(\gamma)$.

Define the width of a line $g \cdot N$ in X_G to be $n(n-1)$ if $g \cdot N \cap \gamma = n$. The width here is chosen so that perturbing γ to a generic projection which has only double point singularities preserves the width. Only crossings contribute to the width, which is equal to the crossing number of the projection of γ to the xy -plane. The 2-width of a knot is equal to the knot's crossing number.

Example 4. Let M, N, G be \mathbb{R}^3 , the xy -plane and \mathbb{R}^1 respectively as in Example 1. But we now allow non-transverse intersections and take the width of $\{g(N) : g \in G\}$ to be the intersection number of Γ and $\{g(N) : g \in G\}$ counted without multiplicity.

For a generic γ , the equivalence classes of X_G consist of

- Two components represented by planes that miss γ , one above and one below γ .
- One component for each equivalence class intersecting γ transversely.
- One component for each plane intersecting γ non-transversely at one or more points.

The width minimizing representative of an n -bridge knot is a curve γ in bridge position, with all its maximums at one level and all its minimum at a second level. A width-minimizing curve has $w_G = 4n$ where n is the bridge number.

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